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Conical geometry and quantum entropy of a charged Kerr black hole

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Abstract

We apply the method of conical singularities to calculate the tree-level entropy and its one-loop quantum corrections for a charged Kerr black hole. The Euclidean geometry for the Kerr-Newman metric is considered. We show that for an arbitrary periodization in Euclidean space there exists a conical singularity at the horizon. Its δ-function like curvatures are calculated and are shown to behave similar to the static case. The heat kernel expansion for a scalar field on this conical space background is derived and the (divergent) quantum correction to the entropy is obtained. It is argued that these divergences can be removed by renormalization of couplings in the tree-level gravitational action in a manner similar to that for a static black hole.

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1 Introduction

The notion that black holes could be considered as a thermodynamic systems characterized by temperature, energy and entropy was first proposed by Bekenstein [1] and confirmed via the discovery of their thermal radiation properties by Hawking [2]. Independently, it was realized that there are only a few macroscopic parameters which can be assigned to a black hole: its mass (m), charge (q) and angular momentum (Ω). In the static case, angular momentum vanishes. A typical representative of this class is the Reissner-Nordstrom black hole which is a solution of the Einstein equations with a Maxwell field as a source. Such a hole is characterized by just its mass (m) and charge (q).

When rotation is present, the Einstein-Maxwell equations have the Kerr-Newman metric [3] as a solution. This metric corresponds to a black hole of a general type characterized by all three parameters (m, q, Ω). Remarkably, the thermodynamic analogy works for this general case; in particular, it suggests that there is an entropy associated with this hole that is proportional to the area of the horizon. If this analogy is exact, there must be hidden degrees of freedom of the hole which are counted by the Bekenstein-Hawking entropy. Recently, there has been much interest in attempting to provide a statistical explanation of these degrees of freedom and their relationship to the entropy (see reviews [4], [5]) within some quantum-mechanical calculations [6]-[21]. However, the proposed expressions for the entropy can be considered to be quantum (one-loop) corrections to the classical quantity, and do not give any explanation of the classical entropy itself.

According to 't Hooft [6], one can relate the entropy of a black hole with a thermal gas of quantum field excitations propagating outside the horizon. In his model 't Hooft introduced a "brick wall" cut-off: a fixed boundary near the horizon within which the quantum field does not propagate. Its role is to eliminate divergences which appear due to the infinite growth of the density of states close to the horizon. This model can be successfully formulated in different space-time dimensions [7]. The quantization of a field system typically requires an ultraviolet (UV) regularization procedure that must be taken into account in the statistical-mechanical calculation as well. Remarkably, it was demonstrated in [8] that the Pauli-Villars regularization not only removes the standard

field-theoretical UV-divergences but automatically implements a cut-off in the 't Hooft calculation, rendering unnecessary the introduction of the "brick wall".

The natural way to formulate black hole thermodynamics is to use the Euclidean path integral approach [22]. For an arbitrary field system it entails closing the Euclidean time coordinate with a period $\beta = T^{-1}$ where T is the temperature of the system. This yields a periodicity condition for the quantum fields in the path integral. In the black hole case for arbitrary β this procedure leads to an effective Euclidean manifold which has a conical singularity at the horizon that vanishes for a fixed value $\beta = \beta_H$. Thermodynamic quantities (*i.e.* energy and entropy) are calculated by differentiating the corresponding free energy with respect to β and then setting $\beta = \beta_H$. This procedure was consistently carried out for a static black hole and resulted in obtaining the general UV-divergent structure of the entropy [18], [19], [20], [21].

Essentially, the divergences of the entropy have the same origin as the UV-divergences of the quantum effective action and can be removed by remormalization of the gravitational couplings in the tree-level gravitational action [16], [17], [21]. The technique developed in [21] allowed proof of this statement for an arbitrary static black hole. Alternatively, this was demonstrated for the Reissner-Nordstrom black hole within 't Hooft's approach [8] applying the Pauli-Villars regularization scheme.

An essential loophole in the above considerations is that they were concerned with only static, non-rotating black holes. The only exception is a series of recent preprints [23] where 't Hooft's approach was applied to a Kerr-Newman black hole and some (qualitative) analysis of divergences was presented. Adoption of conical methods for stationary black holes necessitates dealing with problems of treating Euclideanization (or complexification) of the Kerr-Newman metric [24] and a general periodicity analysis of its conical geometry. Although the passage to a Euclidean metric and periodicity arguments were given some time ago [22] the conical geometry for arbitrary period in the Kerr-Newman case remains unclear. Additional outstanding questions include the structure of the UV-divergences of the entropy of stationary black holes and whether or not their renormalization works in the same way as for a static hole.

In this paper we consider these questions in detail. In Section 2 we describe the passage

to Euclidean space for the Kerr-Newman metric, establishing the structure of this space in the vicinity of the horizon. We determine the conditions necessary for periodicity in the direction of the time-like Killing vector (which is analog of the Euclidean time vector ∂_τ in the static case) corresponding to regular Euclidean space. For arbitrary periods there is a conical singularity at the horizon surface. The geometry of this conical space is studied in Section 3. We employ the regularization method suggested in [25] and obtain the expected δ-function like behavior of the curvatures. The integrals of quadratic combinations of curvature tensors are also considered. The results we obtain have a marked similarity to the static case. In Section 4 we consider the Euclidean path integral quantization of a scalar matter field in the background of a conical Kerr-Newman metric. We obtain the UV-divergences for the entropy, the structure of which is similar to that obtained in the static case. We argue that these divergences of entropy are renormalized in the same way as for a static black hole.

2 Euclidean Kerr-Newman geometry

The Kerr-Newman metric of the space-time with Minkowskian signature in Boyer-Lindquist coordinates takes the form:

$$
\begin{aligned}
& ds^2 = g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \\
& g_{rr} = \frac{\rho^2}{\Delta} \ , \quad g_{\theta\theta} = \rho^2 \ , \quad g_{tt} = -\frac{(\Delta - a^2\sin^2\theta)}{\rho^2} \ ; \\
& g_{t\phi} = -\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2} \ , \quad g_{\phi\phi} = \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta \ , \\
& \Delta(r) = r^2 + a^2 + q^2 - 2mr \ , \quad \rho^2 = r^2 + a^2\cos^2\theta
\end{aligned} \tag{2.1}
$$

The function $\Delta(r)$ can be represented in the form $\Delta(r) = (r-r_+)(r-r_-)$, where $r_\pm = m \pm \sqrt{m^2-a^2-q^2}$.

This space-time has a pair of orthogonal Killing vectors:

$$
\begin{aligned}
& K = \partial_t + \frac{a}{r^2+a^2}\partial_\phi \ , \quad \tilde{K} = a\sin^2\theta\partial_t + \partial_\phi \\
& K^2 = -\frac{\Delta\rho^2}{(r^2+a^2)^2} \ , \quad \tilde{K}^2 = \rho^2\sin^2\theta \ , \quad K\cdot\tilde{K} = 0 \ .
\end{aligned} \tag{2.2}
$$

The vector K is time-like everywhere in the region $r \geq r_+$ and becomes null $K^2 = 0$ for $r = r_+$, whereas $\tilde{K}$ is space-like everywhere outside the axis ($\theta = 0,\ \theta = \pi$) where $\tilde{K}^2 = 0$.

The one-forms dual to K and $\tilde{K}$ are

$$\begin{aligned}
\omega &= \frac{(r^2+a^2)}{\rho^2}(dt - a\sin^2\theta d\phi) \\
\tilde{\omega} &= \frac{(r^2+a^2)}{\rho^2}(d\phi - \frac{a}{(r^2+a^2)}dt) \\
\omega[K] &= \tilde{\omega}[\tilde{K}] = 1\ , \quad \omega[\tilde{K}] = \tilde{\omega}[K] = 0
\end{aligned} \tag{2.3}$$

To obtain the correspondence with the Schwarzschild metric note that K and $\tilde{K}$ are the respective analogs of the vectors ∂_t and ∂_ϕ and ω and $\tilde{\omega}$ are the respective analogs of dt and $d\phi$ of the Schwarzschild metric. This correspondence is almost exact with one exception: ω and $\tilde{\omega}$ together with $d\theta$ and dr form an anholonomic basis of one-forms. This means that there are no globally defined coordinates X and $\tilde{X}$ such that $\omega = dX$ and $\tilde{\omega} = d\tilde{X}$.

Horizon surfaces Σ are defined as the surfaces where the time-like vector K becomes null, $K^2|_\Sigma = 0$; the outer horizon is the surface for which $r = r_+$. In addition to this one often considers the surface where the vector ∂_t becomes null. This surface is called the ergosphere and is determined by equation $r^2 + a^2\cos^2\theta + q^2 - 2mr = 0$. It lies outside the outer horizon Σ, touching it at the axis $\theta = 0$ and $\theta = \pi$.

Consider now the Euclideanization of the Kerr-Newman metric. The standard prescription says [22] that we must change the time variable $t = \imath\tau$ and supplement this by the parameter transformation $a = \imath\hat{a}$, $q = \imath\hat{q}$. The *Euclidean* vectors K, $\tilde{K}$ and the corresponding one-forms ω, $\tilde{\omega}$ take the form:

$$\begin{aligned}
K &= \partial_\tau - \frac{\hat{a}}{r^2-\hat{a}^2}\partial_\phi\ , \quad \tilde{K} = \hat{a}\sin^2\theta\partial_\tau + \partial_\phi \\
\omega &= \frac{(r^2-\hat{a}^2)}{\hat{\rho}^2}(d\tau - \hat{a}\sin^2\theta d\phi) \\
\tilde{\omega} &= \frac{(r^2-\hat{a}^2)}{\hat{\rho}^2}(d\phi + \frac{\hat{a}}{(r^2-\hat{a}^2)}d\tau)
\end{aligned} \tag{2.4}$$

where $\hat{\rho}^2 = r^2 - \hat{a}^2\cos^2\theta$. The Euclidean metric can be written in the form:

$$ds_E^2 = \frac{\hat{\rho}^2}{\hat{\Delta}}dr^2 + \frac{\hat{\Delta}\hat{\rho}^2}{(r^2-\hat{a}^2)^2}\omega^2 + \hat{\rho}^2(d\theta^2 + \sin^2\theta\tilde{\omega}^2) \tag{2.5}$$

where ω and $\tilde{\omega}$ take the form (2.4), $\hat{\Delta} = r^2 - \hat{a}^2 - \hat{q}^2 - 2mr$. Roots of the function $\hat{\Delta}$ are now $\hat{r}_\pm = m \pm \sqrt{m^2+\hat{a}^2+\hat{q}^2}$. The horizon surface Σ defined by $r = \hat{r}_+$ is the

stationary surface of the Killing vector K. Consider metric (2.5) near $r = \hat{r}_+$. It is useful to introduce a new *radial* variable x such that near the horizon we have

$$\hat{\Delta} = \gamma(r - \hat{r}_+) = \frac{\gamma^2 x^2}{4},$$
$$(r - \hat{r}_+) = \frac{\gamma x^2}{4}, \quad \gamma = 2\sqrt{m^2 + \hat{a}^2 + \hat{q}^2} \tag{2.6}$$

Then the metric (2.5) up to terms $O(x^2)$ reads:

$$ds_E^2 = ds_\Sigma^2 + \hat{\rho}_+^2 \left(dx^2 + \frac{\gamma^2 x^2}{4(\hat{r}_+^2 - \hat{a}^2)^2} \omega^2 \right) \quad , \tag{2.7}$$

where $\hat{\rho}_+^2 = \hat{r}_+^2 - \hat{a}^2 \cos^2\theta$ and

$$ds_\Sigma^2 = \hat{\rho}_+^2 \left(d\theta^2 + \sin^2\theta \tilde{\omega}^2 \right)$$
$$= \hat{\rho}_+^2 d\theta^2 + \frac{(\hat{r}_+^2 - \hat{a}^2)^2}{\hat{\rho}_+^2} \sin^2\theta d\psi^2 \tag{2.8}$$

is metric on the horizon surface Σ. In writing (2.8) we employed the fact that on Σ we may introduce the well-defined angle coordinate $\psi = \phi + \frac{\hat{a}}{(\hat{r}_+^2 - \hat{a}^2)}\tau$. The regularity of the metric (2.8) at the points $\theta = 0, \quad \theta = \pi$ requires the identification of the points ψ and $\psi + 2\pi$ on Σ. After all calculations with the Euclideanized Kerr-Newman have been completed, we analytically continue the results obtained back to the real values a and q.

Expression (2.7) may be rewritten as follows:

$$ds_E^2 = ds_\Sigma^2 + \hat{\rho}_+^2 ds_{C_2}^2 \tag{2.9}$$

where $ds_{C_2}^2$ is metric of a two-dimensional disk C_2 attached to the horizon Σ at a point $(\theta, \ \psi)$:

$$ds_{C_2}^2 = dx^2 + \frac{\gamma^2 x^2}{4\hat{\rho}_+^4}(d\tau - \hat{a}\sin^2\theta d\phi)^2 \quad . \tag{2.10}$$

Confider the metric (2.10) with $(\theta, \ \psi)$ fixed. Then we may introduce an angle coordinate on C_2, $\chi = \tau - \hat{a}\sin^2\theta \ \phi$, in terms of which the metric reads

$$ds_{C_2}^2 = dx^2 + \frac{\gamma^2 x^2}{4\hat{\rho}_+^4} d\chi^2 \quad . \tag{2.11}$$

Requiring the absence of a conical singularity at $x = 0$ means that we must identify points χ and $\chi + 4\pi\gamma^{-1}\hat{\rho}_+^2$. In order for this to hold independently of the coordinate θ on the horizon we must also identify points $(\tau, \ \phi)$ with $(\tau + 2\pi\beta_H, \ \phi - 2\pi\Omega\beta_H)$, where

$\Omega = \frac{\hat{a}}{(\hat{r}_+^2 - \hat{a}^2)}$ is the (complex) angular velocity and $\beta_H = \frac{(\hat{r}_+^2 - \hat{a}^2)}{\sqrt{m^2 + \hat{a}^2 + \hat{q}^2}}$. It is easy to see that the identified points have the same coordinate ψ.

With the described identification we obtain the following picture of the Euclidean Kerr-Newman geometry in the vicinity of the horizon Σ. Attached to every point (θ, ψ) of the horizon is a two-dimensional disk C_2 with coordinates (x, χ). Although χ is not a global coordinate in four-dimensional space and at each point (θ, ψ) there is a new χ, the periodic identification of points on C_2 works universally and independently of any point on the horizon Σ. As in static case, the Euclidean Kerr-Newman geometry possesses an abelian isometry generated by the Killing vector K with horizon surface Σ being a fixed set of the isometry. Globally, K is not a coordinate vector. However, locally we have $K = \partial_\chi$ where χ was introduced above. The periodicity is in the direction of the vector K and the resulting Euclidean space E is regular manifold.

3 Conical singularity and curvature tensors

Assume now that we close the trajectory of the Killing vector K with arbitrary period $2\pi\beta$. Near the horizon this means that on C_2 in (2.9), (2.10) we identify points $(\tau,\ \phi)$ and $(\tau + 2\pi\beta,\ \phi - 2\pi\Omega\beta)$ with $\beta \neq \beta_H$. Note again that points identified in this way have the same value of the coordinate ψ. Then χ is an angle coordinate with period $2\pi\beta(1 + \hat{a}\Omega\sin^2\theta)$. By introducing a new angle coordinate $\chi = \beta\hat{\rho}_+^2(\hat{r}_+^2 - \hat{a}^2)^{-1}\bar{\chi}$ which has period 2π, the metric on C_2 becomes

$$ds^2_{C_{2,\alpha}} = dx^2 + \alpha^2 x^2 d\bar{\chi}^2 \tag{3.1}$$

and coincides with the metric on a two dimensional cone with angular deficit $\delta = 2\pi(1-\alpha)$, $\alpha = \frac{\beta}{\beta_H}$. With the above identification the four-dimensional metric (2.5) describes the Euclidean conical space E_α with singular surface Σ.

Curvature tensors at conical singularities behave as distributions. This behavior was precisely established for a flat 2d cone in [26] and for a general static metric in [25]. The Kerr-Newman metric, which is the subject of our consideration here, is stationary and not static. Therefore, all the formulae obtained in [25] must be checked for this case.

To proceed, we apply the method which was successful in the static case (see details in [25]). It consists in regulating the conical singularity when the cone metric (3.1) is replaced by a sequence of regular metrics labeled by a parameter b:

$$ds^2_{C_{2,\alpha,b}} = f(x,b)dx^2 + \alpha^2 x^2 d\bar{\chi}^2 \quad , \tag{3.2}$$

where $f(x,b)$ is some smooth regulating function that approaches unity as $b \to 0$, *e.g.*

$$f(x,b) = \frac{x^2 + \alpha^2 b^2}{x^2 + b^2} \tag{3.3}$$

is a suitable regularization function. In the limit $b \to 0$ the sequence of metrics (3.2) re-produces a δ-function-like contribution to the curvature.

Applying this method to the Kerr metric consider a small vicinity of the horizon surface Σ. For $\beta \neq \beta_H$ the metric there reads

$$ds^2_{E_\alpha} = ds^2_\Sigma + \hat{\rho}^2_+ ds^2_{C_\alpha} \tag{3.4}$$

Replacing the cone metric $ds^2_{C_\alpha}$ by $ds^2_{C_{\alpha,b}}$ (3.2) we obtain a sequence of regular metrics

$$ds^2_{E_{\alpha,b}} = ds^2_\Sigma + \hat{\rho}^2_+ ds^2_{C_{\alpha,b}} \tag{3.5}$$

To calculate the curvature we define the (anholonomic) basis of one-forms $\{e^a,\ a = 1, ..., 4\}$ orthonormal with respect to metric (3.5):

$$\begin{aligned}
e^1 &= b\hat{\rho}_+ f^{1/2}(y)dy \\
e^2 &= by\frac{(m^2 + \hat{a}^2 + \hat{q}^2)^{1/2}}{\hat{\rho}_+}(d\tau - \hat{a}\sin^2\theta d\phi) \\
e^3 &= \hat{\rho}_+ d\theta \\
e^4 &= \frac{(\hat{r}^2_+ - \hat{a}^2)}{\hat{\rho}_+}\sin\theta(d\phi + \frac{\hat{a}}{(\hat{r}^2_+ - \hat{a}^2)}d\tau)
\end{aligned} \tag{3.6}$$

where we changed variables so that $x = by$, $\ f(y) = \frac{y^2+\alpha^2}{y^2+1}$.

The Lorentz connection one-form $\omega^a_{\ b} = \omega^a_{\ b\ c}e^c$ is found from the equation:

$$de^a + \omega^a_{\ b} \wedge e^b = 0 \tag{3.7}$$

We are interested in those components of the Lorentz connection which are singular in the limit $b \to 0$. Analyzing the expressions de^a for the basis (3.6) we observe that only de^2 contains a singular term:

$$de^2 = \frac{dy}{y} \wedge e^2 + ... = (byf^{1/2}(y))^{-1}e^1 \wedge e^2 + ... \quad , \tag{3.8}$$

where "..." means terms finite in the limit $b \to 0$. It follows from (3.8) that the only singular component of the Lorentz connection is

$$\omega^2{}_1 = (by\hat{\rho}_+ f^{1/2}(y))^{-1} e^2 + ... \tag{3.9}$$

The curvature two-form $R^a{}_b = R^a{}_{b\ c\ d} e^c \wedge e^d$ is defined as follows

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad .$$

Again, the only singular component of $R^a{}_b$ is

$$R^2{}_1 = d\omega^2{}_1 + ... = \frac{1}{2yb^2\hat{\rho}_+^2} \frac{f'_y}{f^2} e^2 \wedge e^1 + ...$$

and so in terms of curvature components the only singular component is

$$R_{2121} = \frac{1}{2yb^2\hat{\rho}_+^2} \frac{f'_y}{f^2} + ... \tag{3.10}$$

Introducing a pair of vectors (see Appendix Eqs.(A.1), (A.2)) $n_a = n^\mu_a \partial_\mu, \ a = 1,2$ orthogonal to the horizon Σ and dual to the one-forms $e^a, \ a = 1,2$ we may re-write this $R_{2121} = \frac{1}{2} R_{\mu\nu\alpha\beta} n^\mu_a n^\nu_b n^\alpha_a n^\beta_b$.

In order to show that the component R_{2121} behaves as a δ-function in the limit $b \to 0$, let us consider the integral

$$I_{D_\epsilon} = \int_{D_\epsilon} R_{2121} \ v(x,\theta,\psi) e^1 \wedge e^2 \wedge e^3 \wedge e^4 \tag{3.11}$$

over a small disk D_ϵ surrounding the horizon surface Σ, $0 \leq x \leq \epsilon$. In (3.11) $v(x,\theta,\psi)$ is a test function which is constant along the trajectory of vector K ($K[v] = 0$). It can be expanded as

$$\begin{aligned} v(x,\theta,\psi) &= v_0(\theta,\psi) + v_1(\theta,\psi)x^2 + ... \\ &= v_0(\theta,\psi) + b^2 v_1(\theta,\psi) y^2 + ... \end{aligned}$$

Recall that (θ,ψ) are the coordinates on the horizon Σ. Substituting (3.6), (3.10) into (3.11) we obtain

$$\begin{aligned} I_{D_\epsilon} = \int_0^{\frac{\epsilon}{b}} dy \frac{f'_y}{2f^{3/2}} \sqrt{m^2 + \hat{a}^2 + \hat{q}^2} \oint (d\tau - \hat{a}\sin^2\theta d\phi) \\ \int_\Sigma \frac{1}{\hat{\rho}_+^2} (v_0 + v_1 b^2 y^2 + ...) e^3 \wedge e^4 \end{aligned} \tag{3.12}$$

In (3.12) we first integrate over $e^1 \wedge e^2$ in the subspace orthogonal to Σ under fixed (θ, ψ) and then take the integral $\int_\Sigma e^3 \wedge e^4$ over the horizon. This yields

$$\oint (d\tau - \hat{a}\sin^2\theta d\phi) = \frac{2\pi\beta\hat{\rho}_+^2}{\hat{r}_+^2 - \hat{a}_+^2} \quad , \tag{3.13}$$

where the integration is taken over the closed integral trajectory of the Killing vector K under fixed $(\theta,\ \psi)$.

In the limit $b \to 0$, we have $\frac{\epsilon}{b} \to \infty$ in the y-integration in (3.12) and so obtain

$$\int_0^\infty dy \frac{f'_y}{2f^{3/2}} = -f^{-1/2}(y)|_0^\infty = \frac{1-\alpha}{\alpha} \tag{3.14}$$

Taking into account that $\beta_H = \frac{\hat{r}_+^2 - \hat{a}^2}{\sqrt{m^2+\hat{a}^2+\hat{q}^2}}$ from (3.12)-(3.14) we finally obtain in the limit $b \to 0$:

$$I_{D_\epsilon} = 2\pi(1-\alpha)\int_\Sigma v_0(\theta,\psi) e^3 \wedge e^4 \tag{3.15}$$

Since this holds for arbitrarily small ϵ we conclude that in the limit $b \to 0$ the quantity R_{2121} behaves as a δ-function having support at the surface Σ. Noting that the vectors $n_a,\ a = 1, 2$ introduced above are normal to Σ we may write

$$\begin{aligned}
&R^{\mu\nu}{}_{\alpha\beta} = \bar{R}^{\mu\nu}{}_{\alpha\beta} + 2\pi(1-\alpha)\left((n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha)\right)\delta_\Sigma \\
&R^\mu{}_\nu = \bar{R}^\mu{}_\nu + 2\pi(1-\alpha)(n^\mu n_\nu)\delta_\Sigma \\
&R = \bar{R} + 4\pi(1-\alpha)\delta_\Sigma
\end{aligned} \tag{3.16}$$

where δ_Σ is the delta-function $\int_{\mathcal{M}} f\delta_\Sigma e^1 \wedge e^2 \wedge e^3 \wedge e^4 = \int_\Sigma f e^3 \wedge e^4$ and we denote $(n_\mu n_\nu) = \sum_{a=1}^2 n_\mu^a n_\nu^a$. In particular, it follows from (3.16) that

$$\int_{E_\alpha} R e^1 \wedge e^2 \wedge e^3 \wedge e^4 = \alpha \int_E \bar{R} e^1 \wedge e^2 \wedge e^3 \wedge e^4 + 4\pi(1-\alpha)A_\Sigma \quad , \tag{3.17}$$

where $A_\Sigma = \int_\Sigma e^3 \wedge e^4$ is area of Σ. For the particular case of the Kerr-Newman metric, $\bar{R} = 0$. Remarkably, expressions (3.16), (3.17) are exactly the same as that obtained in [21] for a static metric.

For a variety of applications it is necessary to know the integrals of quadratic combinations of curvatures over the space E_α with a conical singularity at the surface Σ. According to (3.16) curvature $\mathcal{R}$ contains $(1-\alpha)\delta_\Sigma$-contribution as well as a regular part $\bar{\mathcal{R}}$. Therefore, one can expect the result which can be symbolically written as follows

$$\int_{E_\alpha} \mathcal{R}^2 = \alpha \int_{E_{\alpha=1}} \bar{\mathcal{R}}^2 + 2(1-\alpha)\int_\Sigma \bar{\mathcal{R}}_n + O((1-\alpha)^2) \quad , \tag{3.18}$$

where $\bar{\mathcal{R}}_n$ means projection of the tensor $\bar{\mathcal{R}}$ onto the subspace orthogonal to the singular surface Σ. The expression (3.18) is ill-defined since $\mathcal{R}^2$ contains term $((1-\alpha)\delta_\Sigma)^2$. However, it is of higher order with respect to $(1-\alpha)$ and so can be collected in the last term in (3.18).

The form (3.18) was obtained in [25] for a static metric. To verify this for the Kerr-Newman case we must write the metric (2.5) near the horizon Σ, including all terms of order x^2. Taking into account the regularization function $f(x,b)$ as above the metric reads:

$$
\begin{aligned}
ds^2_{E_{\alpha,b}} &= byf(y)dy^2 + \frac{\gamma^2 b^2 y^2}{4\hat{\rho}_+^2}(d\tau - \hat{a}\sin^2\theta d\phi)^2 + (\hat{\rho}_+^2 + \frac{\gamma}{2}b^2y^2\hat{r}_+)d\theta^2 \\
&+ \left(\frac{(\hat{r}_+^2-\hat{a}^2)^2}{\hat{\rho}_+^2} + \frac{(\hat{r}_+^2-\hat{a}^2)}{\hat{\rho}_+^2}(1 - \frac{(\hat{r}_+^2-\hat{a}^2)}{2\hat{\rho}_+^2})\gamma\hat{r}_+b^2y^2\right)\sin^2\theta(d\phi + \frac{\hat{a}}{\hat{r}_+^2-\hat{a}^2}d\tau)^2 \\
&- \frac{\gamma\hat{a}b^2y^2}{2\hat{\rho}_+^2}\sin^2\theta(d\phi + \frac{\hat{a}}{\hat{r}_+^2-\hat{a}^2}d\tau)d\tau
\end{aligned}
\tag{3.19}
$$

The general structure of quadratic combinations of curvature terms (denoted by $\mathcal{R}^2$) for the metric (3.19) symbolically is

$$
\mathcal{R}^2 = b^2 A + \frac{f'_y}{b^2}B + \frac{(f'_y)^2}{b^4}C + O(b^4)
\tag{3.20}
$$

where A, B, C are some functions that are independent of b and do not contain derivatives of the regularization function $f(y)$.

Since the measure of integration in the region near Σ is proportional to b^2 we conclude that second and third terms in (3.20) after integration produce in the limit $b\to 0$ the respective second and third terms in (3.18). In order to get this we use the fact that the derivatives of $f(y)$ behave as $f'(y)\sim(1-\alpha)$.

After straightforward but tedious calculations we obtain in the limit $b\to 0$:

$$
\int_{E_\alpha} R^{\mu\nu}R_{\mu\nu} = \alpha\int_E \bar{R}^{\mu\nu}\bar{R}_{\mu\nu} + 4\pi(1-\alpha)\int_\Sigma \bar{R}_{aa} + O((1-\alpha)^2) \quad ,
\tag{3.21}
$$

$$
\int_{E_\alpha} R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \alpha\int_E \bar{R}^{\mu\nu\lambda\rho}\bar{R}_{\mu\nu\lambda\rho} + 8\pi(1-\alpha)\int_\Sigma \bar{R}_{abab} + O((1-\alpha)^2) \quad ,
\tag{3.22}
$$

where $\bar{R}_{aa} = \sum_{a=1,2}\bar{R}_{\mu\nu}n^\mu_a n^\nu_a$ and $\bar{R}_{abab} = \sum_{a,b=1,2}\bar{R}_{\mu\nu\lambda\rho}n^\mu_a n^\lambda_a n^\nu_b n^\rho_b$.

In obtaining (3.21), (3.22) we made use of the fact (as in (3.12)) that near Σ the measure of integration $\mu_{E_{\alpha,b}}$ takes the form (see (2.5)) $\mu_{E_{\alpha,b}} = \hat{\rho}_+^2\mu_\Sigma\mu_{C_{\alpha,b}}$, where $\mu_\Sigma =$

$(\hat{r}_+^2 - \hat{a}^2)\sin\theta d\theta d\psi$ $(0 \le \psi \le 2\pi)$ is the measure on Σ and $\mu_{C_{\alpha,b}} = \alpha b^2 f^{1/2}(y) dy d\bar{\chi}$ $(0 \le \bar{\chi} \le 2\pi)$ is the measure on the regularised cone $C_{\alpha,b}$. For the integral of R^2 we obtain

$$\int_{E_\alpha} R^2 = O((1-\alpha)^2)$$

in agreement with the expected formula

$$\int_{E_\alpha} R^2 = \alpha \int_E \bar{R}^2 + 8\pi(1-\alpha)\int_\Sigma \bar{R} + O((1-\alpha)^2) \quad , \tag{3.23}$$

since the Kerr-Newman metric satisfies $\bar{R} = 0$.

Again we obtain for a stationary metric with a conical singularity the same expressions (3.21)-(3.23) as for the static case [25].

For the Kerr-Newman metric we have on the horizon Σ:

$$\begin{aligned} \frac{1}{2}\bar{R}_{abab} &= \bar{R}_{2121} = \frac{\hat{r}_+^2(4\hat{q}^2 + 8m\hat{r}_+) - (\hat{q}^2 + 6m\hat{r}_+)\hat{\rho}^2}{\hat{\rho}_+^6} \\ \frac{1}{2}\bar{R}_{aa} &= \bar{R}_{11} = \bar{R}_{22} = \frac{\hat{q}^2}{\hat{\rho}_+^4} \end{aligned} \tag{3.24}$$

and after integration over Σ we get

$$\begin{aligned} \int_\Sigma \bar{R}_{abab} &= 8\pi \frac{(\hat{r}_+^2 + \hat{q}^2)}{\hat{r}_+^2} + 4\pi \frac{\hat{q}^2}{\hat{r}_+^2}\frac{(\hat{r}_+^2 - \hat{a}^2)}{\hat{a}\hat{r}_+}\ln\left(\frac{\hat{r}_+ + \hat{a}}{\hat{r}_+ - \hat{a}}\right) \\ \int_\Sigma \bar{R}_{aa} &= 4\pi \frac{\hat{q}^2}{\hat{r}_+^2}\left(1 + \frac{(\hat{r}_+^2 - \hat{a}^2)}{2\hat{a}\hat{r}_+}\ln\left(\frac{\hat{r}_+ + \hat{a}}{\hat{r}_+ - \hat{a}}\right)\right) \end{aligned} \tag{3.25}$$

The analytic continuation of these expressions back to real values of the parameters a and q requires the substitution

$$\begin{aligned} &\hat{q}^2 = -q^2, \quad \hat{a}^2 = -a^2, \quad \hat{r}_+ = r_+ \\ &\frac{1}{\hat{a}}\ln\left(\frac{\hat{r}_+ + \hat{a}}{\hat{r}_+ - \hat{a}}\right) = \frac{2}{a}\tan^{-1}\left(\frac{a}{r_+}\right) \end{aligned} \tag{3.26}$$

4 Heat kernel expansion and entropy

In the Euclidean path integral approach to a statistical field system at temperature $T = (2\pi\beta)^{-1}$ one considers the fields which are periodic with respect to imaginary time τ with period $2\pi\beta$. This works well for a static black hole when the metric does not depend on

the time coordinate τ [22]. One then closes the integral curves of the Killing vector ∂_τ with the period $2\pi\beta$.

For a rotating black hole metric we need to close the integral curves of the vector K (2.4). The result of this is that for arbitrary β we obtain the conical space E_α, the geometry of which was described in the previous section. The partition function then reads

$$Z(\beta)=\int[\mathcal{D}\varphi]\exp[-I_E(\varphi,g_{\mu\nu})]\quad, \tag{4.1}$$

where the matter Euclidean action I_E is considered on the space E_α with appropriate boundary (*i.e.* periodicity) conditions imposed on the matter field(s) φ. The contribution to the entropy is

$$S=-(\beta\partial_\beta-1)\ln Z(\beta)|_{\beta=\beta_H} \tag{4.2}$$

Although the Kerr-Newman metric is a solution of the Einstein equations with the matter source in form of a Maxwell field, the gravitational action is always modified by higher-order curvature terms due to quantum corrections. Such R^2-terms must be added to the action at the outset with some bare constants $(c_{1,B},c_{2,B},c_{3,B})$ (tree-level) to absorb the one-loop infinities. The bare (tree-level) gravitational action functional thus takes the form

$$W_{gr}=\int\sqrt{g}d^4x\left(-\frac{1}{16\pi G_B}R+c_{1,B}R^2+c_{2,B}R_{\mu\nu}R^{\mu\nu}+c_{3,B}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) \tag{4.3}$$

Of course, we assume in addition to (4.3) a classical matter action which can in principle be rather complicated. The corresponding tree-level entropy can be obtained as a replica of the action (4.3) after introducing the regulated conical singularity and applying formulas (4.1)-(4.2). Using formulas (3.21)-(3.23) of the previous Section we obtain for the tree-level entropy:

$$S(G_B,c_{i,B})=\frac{1}{4G_B}A_\Sigma-\int_\Sigma\left(8\pi c_{1,B}\bar{R}+4\pi c_{2,B}\bar{R}_{aa}+8\pi c_{3,B}\bar{R}_{abab}\right) \tag{4.4}$$

where $\bar{R}_{aa}=\sum_{a=1,2}\bar{R}_{\mu\nu}n^\mu_a n^\nu_a$ and $\bar{R}_{abab}=\sum_{a,b=1,2}\bar{R}_{\mu\nu\lambda\rho}n^\mu_a n^\lambda_a n^\nu_b n^\rho_b$, $\{n^\mu_a,\ a=1,2\}$ are vectors normal to Σ. This is exactly the same expression that we had for the static case [27], [21]. Expression (4.4) is really valid off-shell, as we do not require the metric to

satisfy any equations of motion. On-shell we must substitute in (4.4) the field equation $\bar{R} = 0$ satisfied by the Kerr-Newman metric.

At the one-loop level we consider the matter action in the form

$$I_E = \frac{1}{2} \int_{E_\alpha} (\nabla\varphi)^2$$

and get for the partition function

$$\ln Z(\beta) = -\frac{1}{2} \ln det(-\Box_{E_\alpha})$$

expressed via the determinant of the Laplacian $\Box = \nabla_\mu \nabla^\mu$ over the conical space E_α. In the De Witt-Schwinger proper time representation we have for the logarithm of the determinant:

$$\ln det(-\Box) = -\int_{\epsilon^2}^{\infty} \frac{ds}{s} Tr(e^{s\Box}) \tag{4.5}$$

In four dimensions we have the asymptotic expansion

$$Tr(e^{s\Box}) = \frac{1}{(4\pi s)^2} \sum_{n=0}^{\infty} a_n s^n \tag{4.6}$$

and for the divergent part of $(\ln Z)_{div}$ we get

$$(\ln Z)_{div} = \frac{1}{32\pi^2} (\frac{1}{2} a_0 \epsilon^{-4} + a_1 \epsilon^{-2} + 2a_2 \ln \frac{L}{\epsilon}) \quad , \tag{4.7}$$

where L is an infra-red cut-off. It is known that for a manifold with conical singularities the heat kernel coefficients in (4.6) are really a sum

$$a_n = a_n^{st} + a_{n,\alpha} \tag{4.8}$$

of standard plus conical coefficients. The standard coefficients $\bar{a}_n^{st}$ are the same as for for a smooth manifold E [28]:

$$\begin{aligned} a_0^{st} &= \int_{E_\alpha} 1 \quad , \quad a_1^{st} = \frac{1}{6} \int_{E_\alpha} \bar{R} \\ a_2^{st} &= \int_{E_\alpha} \left(\frac{1}{180} \bar{R}_{\mu\nu\alpha\beta} \bar{R}^{\mu\nu\alpha\beta} - \frac{1}{180} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} - \frac{1}{30} \Box \bar{R} + \frac{1}{72} \bar{R}^2 \right) \end{aligned} \tag{4.9}$$

whereas the parts coming from the singular surface Σ (stationary point of the isometry) are

$$\begin{aligned} & a_{0,\alpha} = 0; \quad a_{1,\alpha} = \frac{\pi}{3} \frac{(1-\alpha^2)}{\alpha} \int_\Sigma \sqrt{\gamma} d^2\theta \; ; \\ & a_{2,\alpha} = \frac{\pi}{3} \frac{(1-\alpha^2)}{\alpha} \int_\Sigma (\frac{1}{6} \bar{R} + \lambda_1 (\kappa^a \kappa^a - 2tr(\kappa.\kappa))) \sqrt{\gamma} d^2\theta \\ & -\frac{\pi}{180} \frac{(1-\alpha^4)}{\alpha^3} \int_\Sigma (\bar{R}_{aa} - 2\bar{R}_{abab} + \frac{1}{2} \kappa^a \kappa^a + \lambda_2 (\kappa^a \kappa^a - 2tr(\kappa.\kappa))) \sqrt{\gamma} d^2\theta \end{aligned} \tag{4.10}$$

where $\lambda_{1,2}$ are some constants and $\kappa^a_{\mu\nu},\ a=1,2$ is the extrinsic curvature of the surface Σ with respect to normal vector $n_a,\ a=1,2$; $\kappa^a = g^{\mu\nu}\kappa^a_{\mu\nu}$, $tr(\kappa.\kappa)=\sum_{a=1,2}\kappa^a_{\mu\nu}\kappa^{\mu\nu}_a$.

The expression (4.10) for some special spaces has been known for some time [29]. For a general static metric it was derived recently by Fursaev [30], in the case that all extrinsic curvatures $\kappa^a_{\mu\nu}$ vanished. Dowker [31] derived the heat kernel coefficients in the form (4.10) for an arbitrary conical metric of a general type with a surface Σ having non-trivial extrinsic geometry. Very general arguments were used in [31] to derive the general structure of (4.10): $O(2)$-invariance, dimensionality and conformal invariance. The result (4.10) contains some unknown constants λ_1 and λ_2 in front of the conformal-invariant combination $(\kappa^a\kappa^a - 2tr(\kappa.\kappa))$. The analysis of [31] does not provide a prescription for obtaining the explicit values for these constants.

Applying the formula (4.2) to (4.7) and taking into account that the standard coefficients $a^{st}_n \sim \alpha$ we obtain for the divergent quantum correction to the entropy

$$\begin{aligned} S_{div} = &\frac{1}{48\pi\epsilon^2}A_\Sigma + (\frac{1}{144\pi}\int_\Sigma \bar{R} - \frac{1}{16\pi}\frac{1}{45}\int_\Sigma(\bar{R}_{aa} - 2\bar{R}_{abab}) \\ &-\frac{1}{16\pi}\frac{1}{90}\int_\Sigma \kappa^a\kappa^a + \frac{1}{24\pi}(\lambda_1 - \frac{\lambda_2}{30})\int_\Sigma(\kappa^a\kappa^a - 2tr(\kappa.\kappa)))\ln\frac{L}{\epsilon} \quad . \end{aligned} \tag{4.11}$$

We see that the divergent part of the entropy (4.11) depends both on the projections of the curvatures, $\bar{R}_{aa}$ and $\bar{R}_{abab}$, onto the subspace normal to the horizon surface Σ and on the quadratic combinations of the extrinsic curvatures of Σ. For the static case all extrinsic curvatures vanish and (4.11) repeats the form of the tree-level entropy (4.4). This fact allows one to prove [21] for arbitrary static black holes the statement [16]-[18] that all the UV divergences of entropy are absorbed in the standard renormalization of the gravitational couplings (G, c_i) in the tree-level gravitational action (4.3). Applying the same line of reasoning to the Kerr-Newman black hole entails studying the external geometry of the horizon Σ of the charged rotating black hole. For this case we find that $\sum_{a=1,2}\kappa^a\kappa^a = tr(\kappa.\kappa) = 0$ (see Appendix). This makes the coefficients (4.10) and the expression for S_{div} (4.11) for the Kerr-Newman metric the same as for a static metric.

Consequently, S_{div} in (4.11) repeats the form of the tree-level entropy and the *renormalization* statement is valid for a stationary hole as well. In one sense this is not surprising since the classical thermodynamics of static and stationary holes are formu-

lated in the same way. One could therefore expect this to also be valid in the quantum case.

Consider (4.11) on the Kerr-Newman background. Substituting here (3.25), the condition $\bar{R}=0$, and making the analytic continuation (3.26), we finally obtain for the quantum entropy of the Kerr-Newman black hole:

$$S_{div}=\frac{1}{48\pi\epsilon^2}A_\Sigma+\frac{1}{45}\left(1-\frac{3q^2}{4r_+^2}(1+\frac{(r_+^2+a^2)}{ar_+}\tan^{-1}(\frac{a}{r_+}))\right)\ln\frac{L}{\epsilon} \tag{4.12}$$

where $A_\Sigma=4\pi(r_+^2+a^2)$ is area of the horizon Σ. In the limit $a\to 0$ this expression reduces to that of the Reissner-Nordstrom black hole obtained in [20] using the conical method and in [8] within the framework of a statistical-mechanical calculation in spirit of t'Hooft's approach. Surprisingly, in the uncharged case ($q=0$) the second term in the expression (4.12) does not depend on the rotation parameter a and it is the same as for the Schwarzchild black hole [18]. We do not have an explanation of this fact.

5 Conclusions

The Euclidean approach to black hole thermodynamics implying the conical singularity method is known to be very useful in the static case. It allows one to get both the classical and quantum thermodynamic quantities of static black holes. We have proposed that the thermodynamics of the classical static and stationary black holes are formulated in a similar way. The underlying assumption is that the conical singularity technique can be applied to the rotating hole as well.

In this paper we logically followed this line of reasoning. We studied the Euclidean geometry of the Kerr-Newman metric for an arbitrary period along the time-like Killing vector generating the abelian isometry of the space. The conical geometry of the space near the horizon was established and the δ-function like behavior of the curvatures obtained. This behavior strongly resembles that of a static black hole.

The essential point of formulating the quantum thermodynamics of static black hole is the proving the statement that all the UV-divergences of the entropy of black hole due to quantum matter are removed by the standard renormalization of couplings in the tree-level gravitational action. This allows one to consider the entropy as well-defined

quantum field theoretical quantity. We demonstrate for the Kerr-Newman black hole that S_{div} being expressed via geometrical invariants repeats the form of the tree-level entropy in the same way as for a static case. This proves that the *renormalization* statement works universaly both for the static and stationary holes providing the correct treatment of the quantum thermodynamics.

However, it is still an open question as to what degrees of freedom are counted by the entropy of black hole. A useful approach to this problem is to compare our result with the statistical-mechanical calculation of the quantum entropy of Kerr-Newman black hole along the lines of [6]-[8]. For a charged non-rotating black hole it is known that there is perfect agreement between these two methods (see [8] and [20]). Checking this for stationary case[§] should provide us with a better understanding of the relationship between the different entropies assigned to a black hole [32].

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Appendix: Extrinsic geometry of horizon

With respect to the Euclidean metric (2.5) we may define a pair of orthonormal vectors $\{n_a = n_a^\mu \partial_\mu \ , \quad a = 1, 2\}$:

$$n_1^r = \sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \tag{A.1}$$

$$n_2^\tau = \frac{(r^2 - \hat{a}^2)}{\sqrt{\hat{\Delta}\hat{\rho}^2}} \ , \quad n_2^\phi = \frac{-\hat{a}}{\sqrt{\hat{\Delta}\hat{\rho}^2}} \tag{A.2}$$

Covariantly these are

$$n_r^1 = \sqrt{\frac{\hat{\rho}^2}{\hat{\Delta}}} \tag{A.3}$$

$$n_\tau^2 = \sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \ , \quad n_\phi^2 = -\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \hat{a} \sin^2\theta \tag{A.4}$$

[§]The recent statistical calculation performed in [23] appears to be unsatisfactory since it relates the entropy of rotating hole with data on the ergosphere rather than on the horizon.

The vectors n^1 and n^2 are normal to the horizon surface Σ (defined as $r = r_+,\ \Delta(r = r_+) = 0$), which is a two-dimensional surface with induced metric $\gamma_{\mu\nu} = g_{\mu\nu} - n^1_\mu n^1_\nu - n^2_\mu n^2_\nu$. The (non-zero) components of the induced metric are

$$\begin{aligned} \gamma_{\theta\theta} &= \rho^2, \quad \gamma_{\tau\tau} = \frac{\hat{a}^2 \sin^2\theta}{\hat{\rho}^2} \\ \gamma_{\tau\phi} &= \frac{\hat{a}(r^2 - \hat{a}^2)\sin^2\theta}{\hat{\rho}^2} \\ \gamma_{\phi\phi} &= \frac{(r^2 - \hat{a}^2)^2 \sin^2\theta}{\hat{\rho}^2} \end{aligned} \tag{A.5}$$

With respect to the normal vectors $n^a,\ a = 1, 2$ we may define [33] the extrinsic curvatures of the surface Σ: $\kappa^a_{\mu\nu} = -\gamma^\alpha_\mu \gamma^\beta_\nu \nabla_\alpha n^a_\beta$. We find

$$\begin{aligned} \kappa^1_{\theta\theta} &= -r\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \\ \kappa^1_{\tau\tau} &= \frac{r\hat{a}^2 \sin^2\theta}{\hat{\rho}^4}\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \\ \kappa^1_{\tau\phi} &= -\frac{\hat{a}r(r^2 - \hat{a}^2)\sin^2\theta}{\hat{\rho}^4}\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \\ \kappa^1_{\phi\phi} &= -\frac{r(r^2 - \hat{a}^2)^2\sin^2\theta}{\hat{\rho}^4}\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} \kappa^2_{r\tau} &= -\frac{\hat{a}^2 \sin\theta\cos\theta}{\hat{\rho}^2}\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \\ \kappa^2_{r\phi} &= -\frac{\hat{a}(r^2 - \hat{a}^2)\sin\theta\cos\theta}{\hat{\rho}^2}\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}} \end{aligned} \tag{A.7}$$

For the trace of the extrinsic curvatures, $\kappa^a = \kappa^a_{\mu\nu} g^{\mu\nu}$, we obtain:

$$\kappa^1 = -\frac{2r}{\hat{\rho}^2}\sqrt{\frac{\hat{\Delta}}{\hat{\rho}^2}}\ , \quad \kappa^2 = 0 \tag{A.8}$$

which clearly vanishes when restricted to the surface Σ ($\Delta(r = \hat{r}_+) = 0$).

The quadratic combinations

$$\begin{aligned} \kappa^1_{\mu\nu}\kappa^{\mu\nu}_1 &= \frac{2r^2\hat{\Delta}}{\hat{\rho}^6} \\ \kappa^2_{\mu\nu}\kappa^{\mu\nu}_2 &= \frac{2\hat{a}^2\cos^2\theta\hat{\Delta}}{\hat{\rho}^6} \end{aligned} \tag{A.9}$$

vanish Σ separately both in the static ($a = 0$) and stationary ($a \neq 0$) cases. Consequently, we have $tr(\kappa.\kappa) = \kappa^a_{\mu\nu}\kappa^{a\mu\nu} = 0$ on the horizon.

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